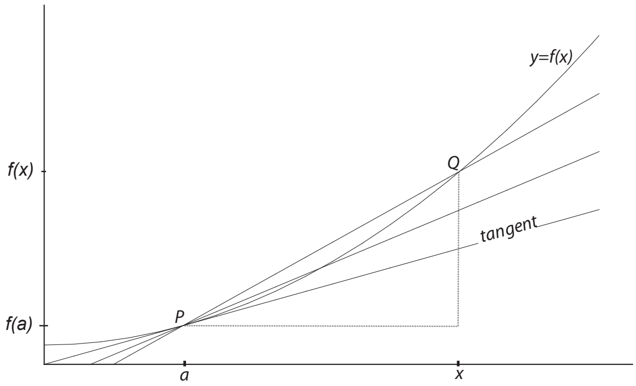
Part 3.1 Differentiation v2 2019-20

Definition



Consider a "nice, smooth" function f, such as the one above, with a fixed point $P = (a, f(a))$. The slope, or gradient, of the chord from P to another point $Q = (x, f(x))$ on the curve is given by

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}.$$

As Q gets "closer and closer" to P, the sequence of chords "seems" to be getting "closer and closer" to a fixed straight line, the *tangent* of f at a. The gradient of the tangent at a, if it exists, will be the derivative of f at a.

Definition 3.1.1 *(Cauchy 1821) Let $f : A \to \mathbb{R}$ and suppose that A contains a neighbourhood of a. We say that f is* ***differentiable at*** *a if, and only if,*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. The value of this limit is the ***derivative*** *of f at a, and is denoted by $f'(a)$.*

(Recall our conventions concerning limits; to say a limit *exists* is to assume that it is finite.)

In the definition we could have written $x = a + h$, and noted that $x \to a$ if, and only if, $h \to 0$. Thus we also have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

A function is differentiable ***on an open interval*** if it is differentiable *at every point in* that interval.

Where defined, $f'(x)$ is a function of the variable x. If we set $y = f(x)$ then sometimes we write

$$\frac{dy}{dx}, \quad \text{or even} \quad \frac{dy}{dx}(x), \quad \text{instead of } f'(x),$$

and

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \frac{dy}{dx}(a) \quad \text{instead of } f'(a).$$

Example 3.1.2 *Using any results about limits that you feel appropriate show that for $n \in \mathbb{N}$ the function*

$$f : \mathbb{R} \to \mathbb{R}, \ x \mapsto x^n$$

is differentiable for all $x \in \mathbb{R}$ and find it's derivative.

Solution Let $n \geq 1$ and $a \in \mathbb{R}$ be given. Consider, for $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}.$$

Polynomials are everywhere continuous so the value of the limit at a is the value of the polynomial and so

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = na^{n-1}.$$

Since the limit exists f is differentiable at a with derivative $f'(a) = na^{n-1}$.

Yet a and n were arbitrary and so, for all $n \geq 1$, f is everywhere differentiable with $f'(x) = nx^{n-1}$. ■

Alternatively, consider

$$\frac{f(a+h) - f(a)}{h} = \frac{1}{h}\left((a+h)^n - a^n\right)$$

and apply the Binomial Theorem.

Example 3.1.3 *Extend the above to show that for $n \in \mathbb{N}$ the function*

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \, x \mapsto x^{-n}$$

is differentiable for all $x \in \mathbb{R} \setminus \{0\}$.

Solution left to students (and Tutorial). ■

Example 3.1.4 *Assume that* $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$ *for all* $\alpha, \beta \in \mathbb{R}$. *Prove that*

$$\frac{de^x}{dx} = e^x$$

for all $x \in \mathbb{R}$.

Solution Let $a \in \mathbb{R}$ be given. Consider, for $x \neq a$,

$$\frac{f(a+h) - f(a)}{h} = \frac{e^{a+h} - e^a}{h} = e^a \frac{e^h - 1}{h}.$$

The limit as $h \to 0$ of the last factor was seen in an earlier section on Special limits; giving

$$\lim_{x \to a} \frac{f(a+h) - f(a)}{h} = e^a \lim_{x \to a} \frac{e^h - 1}{h} = e^a.$$

Since the limit exists f is differentiable at a with derivative $f'(a) = e^a$.

Yet a were arbitrary and so f is everywhere differentiable with $f'(x) = e^x$. ■

Example 3.1.5 *Assume the addition formula for sine, namely*

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

for all $\alpha, \beta \in \mathbb{R}$. *Prove that*

$$\frac{d}{dx}\sin x = \cos x,$$

for all $x \in \mathbb{R}$.

Solution Let $a \in \mathbb{R}$ be given.

$$\begin{aligned}
\lim_{x \to a} \frac{\sin x - \sin a}{x - a} &= \lim_{h \to 0} \frac{\sin(h + a) - \sin a}{h} \\
&= \lim_{h \to 0} \frac{\sin h \cos a + \sin a \cos h - \sin a}{h} \\
&= \cos a \lim_{h \to 0} \frac{\sin h}{h} + \sin a \lim_{h \to 0} \frac{\cos h - 1}{h} \\
&\qquad \text{by Sum Rule for Limits,} \\
&= \cos a \times 1 + \sin a \times 0, \\
&\qquad \text{by results from Part 1,} \\
&= \cos a.
\end{aligned}$$

So the limit exists and thus $\sin x$ is differentiable at $x = a$ and it's derivative is

$$\left.\frac{d}{dx}\sin x\right|_{x=a} = \cos a.$$

Yet $a \in \mathbb{R}$ was arbitrary, hence

$$\frac{d}{dx}\sin x = \cos x,$$

for all $x \in \mathbb{R}$. ■

See the Appendix for more discussion on this example and how, to avoid a circular argument, we have to **not** use L'Hôpital's Rule to evaluate $\lim_{h\to 0} (\sin h)/h$. To use L'Hôpital's Rule we need to be able to differentiate $\sin x$. Yet to prove we can differentiate $\sin x$ we need, as seen above, to use $\lim_{h\to 0} (\sin h)/h = 1$.

The following result is one you will have also seen in Complex Analysis.

Theorem 3.1.6 *If a function is differentiable at a point then it is continuous at that point.*

Proof Assume f is differentiable at $a \in \mathbb{R}$. Consider

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a),$$

for $x \neq a$. Let $x \to a$. Then, since f is differentiable at a we have

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a),$$

and, in particular, the limit exists. Also $\lim_{x\to a} (x - a) = 0$.

Since both limits exist, we can use the Product Rule for Limits to say

$$\begin{aligned}\lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}(x - a)\right) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a}(x - a) \\ &= f'(a) \times 0 = 0.\end{aligned}$$

Hence $\lim_{x\to a} (f(x) - f(a)) = 0$, i.e. $\lim_{x\to a} f(x) = f(a)$. Thus f is continuous at a. ■

The converse of this result is **not** true, i.e. f continuous at a does not imply f is differentiable at a. To show this we need a counter-example.

Example 3.1.7 *Show that* $f : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$ *is continuous but not differentiable at* $x = 0$.

Solution For the derivative at 0 consider, for $x \neq 0$,

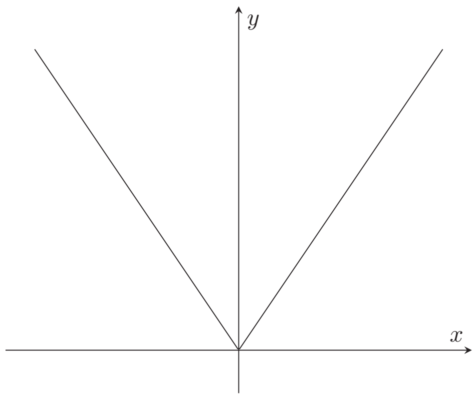
$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \text{sign} x.$$

This is known not to have a limit at 0 (the right hand limit is 1, the left hand limit -1). Hence f is not differentiable at 0. ■

Note The modulus function $|x|$ can be written as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Graphically:



Note In 1877 Weierstrass showed that there exist continuous functions that are nowhere differentiable. See the web site for this course for details of such functions.

Remember The Following:

Differentiable at $a \implies$ Continuous at a
Continuous at $a \not\Longrightarrow$ Differentiable at a

Rules for Differentiation

Since differentiation is defined using limits it can be no surprise that the properties satisfied by derivative should bear a close resemblance to those satisfied by limits. (See Section 1.3.)

Before the next result recall

Lemma 3.1.8 *If f is continuous at a and $f(a) \neq 0$ then there exists $\delta > 0$ such that if $|x-a| < \delta$ then $f(x)$ is non-zero.*

Theorem 3.1.9 ***Rules of Differentiation***

Suppose that both f and g are differentiable at a. Then

***Sum Rule**: $f+g$ is differentiable at a and*

$$(f+g)'(a) = f'(a) + g'(a),$$

***Product Rule**: fg is differentiable at a and*

$$(fg)'(a) = f(a)\,g'(a) + f'(a)\,g(a),$$

***Quotient Rule**: f/g is differentiable at a and*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\,g(a) - f(a)\,g'(a)}{g(a)^2}$$

provided that $g(a) \neq 0$.

Proof of the **Sum Rule** is left to Student.

Product Rule: Consider

$$\begin{aligned}\frac{(fg)(x) - (fg)(a)}{x-a} &= \frac{f(x)\,g(x) - f(a)\,g(a)}{x-a} \\ &= \frac{(f(x) - f(a))\,g(x) + f(a)\,g(x) - f(a)\,g(a)}{x-a} \\ &= g(x)\,\frac{f(x) - f(a)}{x-a} + f(a)\,\frac{g(x) - g(a)}{x-a}\end{aligned}$$

Then

$$\lim_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a}=\lim_{x\to a}g(x)\lim_{x\to a}\frac{f(x)-f(a)}{x-a}+f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a},$$

by the Sum and Product Rules for limits. Allowable since all the limits on the Right Hands Side (RHS) exist. Thus

$$\lim_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a}=g(a)\,f'(a)+f(a)\,g'(a)\,,$$

where $\lim_{x\to a}g(x)=g(a)$ since g is differentiable and so, by Lemma 3.1.6, continuous at a. Since the limit exists fg is differentiable at a with

$$(fg)'(a)=f(a)\,g'(a)+f'(a)\,g(a)\,.$$

Quotient Rule: We are told that g is differentiable at a. This implies that g is continuous at a, i.e. $\lim_{x\to a}g(x)=g(a)$.

By Lemma 3.1.8 because $g(a)\neq 0$ there exists $\delta>0$ such that for $a-\delta<x<a+\delta$ we have $g(x)\neq 0$.

For such x consider

$$\begin{aligned}\frac{\frac{1}{g}(x)-\frac{1}{g}(a)}{x-a} &= \frac{\frac{1}{g(x)}-\frac{1}{g(a)}}{x-a}\\ &= -\frac{1}{g(x)\,g(a)}\frac{g(x)-g(a)}{x-a}.\end{aligned}$$

Now let $x\to a$ to get

$$\lim_{x\to a}\frac{\frac{1}{g}(x)-\frac{1}{g}(a)}{x-a}=-\frac{1}{g(a)}\frac{1}{\lim_{x\to a}g(x)}\lim_{x\to a}\frac{g(x)-g(a)}{x-a},$$

by the Quotient Rule for limits. This is allowable since all the limits on the RHS exist along with $\lim_{x\to a}g(x)=g(a)\neq 0$. Thus

$$\lim_{x\to a}\frac{\frac{1}{g}(x)-\frac{1}{g}(a)}{x-a}=-\frac{g'(a)}{g^2(a)}.$$

Since the limit exists $1/g$ is differentiable at a with

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g^2(a)}.$$

Finally,

$$\left(\frac{f}{g}\right)'(a) = \left(f\frac{1}{g}\right)'(a) = f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\,\frac{1}{g(a)}$$

by the Product Rule. The Quotient Result now follows. ■

Note there is a common mistake made by far too many students attempting to prove the Product Rule. See the Appendix for details.

Example 3.1.10 *All polynomials are differentiable on* $\mathbb{R}$.

Solution is immediate. ■

Theorem 3.1.11 *Rational functions are differentiable wherever they are defined.*

Proof immediate. ■